J Glob Optim (2009) 44:409–421
DOI 10.1007/s10898-008-9339-1

New inexact implicit method for general mixed quasi variational inequalities

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Received: 12 April 2006 / Accepted: 12 August 2008 / Published online: 4 September 2008

Abstract In this paper, we suggest and analyze a new self-adaptive inexact implicit method with a variable parameter for general mixed quasi variational inequalities, where the skew-symmetry of the nonlinear bifunction plays a crucial part in the convergence analysis of this method. We use a self-adaptive technique to adjust parameter ρ at each iteration. The global convergence of the proposed method is proved under some mild conditions. Preliminary numerical results indicate that the self-adaptive adjustment rule is necessary in practice.

Keywords General mixed quasi variational inequalities · Implicit methods · Resolvent operator · Inexact methods · Self-adaptive rules

Mathematics Subject Classification (2000) 49J40 · 65N30

1 Introduction

In recent years variational inequalities have been extended and generalized in different directions. A useful and important generalization of variational inequalities is called general mixed variational inequalities involving the nonlinear bifunction, which unables us to study free, moving, obstacle, equilibrium problems arising in pure and applied sciences in a unified and

Muhammad Aslam Noor is supported by the Higher Education Commission, Pakistan, through research grant No: 1-28/HEC/HRD/2005/90.

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general framework. Due to the presence of the nonlinear bifunction, the projection method and its variant forms including the Wiener–Hopf equations technique can not be extended to suggest iterative methods for solving mixed quasi variational inequalities. To overcome these drawbacks, some iterative methods have been suggested for special cases of the mixed quasi variational inequalities. For example, if the bifunction is proper, convex and lower semicontinuous function with respect to the first argument, then one can show that the mixed quasi variational inequalities are equivalent to the fixed-point problems and the implicit resolvent equations using the resolvent operator technique. This equivalent formulation has been used to suggest and analyze some iterative methods, the convergence of these methods requires that the operator is both strongly monotone and Lipschitz continuous. In recent years, some modified iterative methods [10,13] have been suggested for solving mixed quasi variational inequalities involving the skew-symmetric bifunction. The skew-symmetry of the nonlinear bifunction plays a crucial part in the convergence analysis of these new iterative methods. Noor [8] has used the resolvent operator technique and proposed an implicit method for mixed variational inequalities. But, the numerical experience shows that the method depends significantly on the initial penalty parameter. To overcome this difficulty, Wang et al. [17] have proposed a self-adaptive technique to adjust penalty parameter at each iteration. Recently, Zeng and Yao [18] have presented an inexact implicit method for solving general mixed variational inequalities, they have used the absolute error to control the accuracy of the approximate solution.

Motivated and inspired by the above research, we suggest and analyze an inexact implicit method for mixed quasi variational inequalities. The inexact criteria is more efficient than those in [18]. The numerical experience shows that the method depends significantly on the initial penalty parameter. It converges quite quickly when a proper fixed penalty parameter was chosen. However, this proper penalty parameter is unknown beforehand. If the fixed penalty parameter is large, the number of iterations could be significantly large. Driven by the fact of eliminating this drawback, we have used a self-adaptive technique to adjust penalty parameter at each iteration. The main advantage of this technique can adjust the penalty parameter automatically and the numbers of iterations is insensitive to the different initial penalty parameter.

2 Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, let I be the identity mapping on H, and $T, g : H \to H$ be two operators. Let $\partial\varphi$ denotes the subdifferential of function φ, where $\varphi : H \to R \cup \{+\infty\}$ is a proper convex lower semi continuous function on H. It is well known that the subdifferential $\partial\varphi$ is a maximal monotone operator. We consider the problem of finding $u^* \in H$ such that

$$\langle Tu^*, g(v) - g(u^*)\rangle + \varphi(g(v), g(u^*)) - \varphi(g(u^*), g(u^*)) \geq 0, \quad \forall v \in H, \tag{2.1}$$

where $\varphi(.,.) : H \times H \longrightarrow R \cup \{\infty\}$ is a bifunction continuous with respect to both arguments. Problem (2.1) is called the general mixed quasi variational inequality, see [13] . It has been shown that a wide class of obstacle, unilateral, free, ecology and equilibrium problems can be studied in the framework of mixed quasi variational inequalites (2.1), see [1,2,8,10–18].

For $\varphi(v, u^*) = \varphi(v)$, $\forall u^* \in H$, problem (2.1) reduces to finding $u^* \in H$ such that

$$\langle Tu^*, g(v) - g(u^*)\rangle + \varphi(g(v)) - \varphi(g(u^*)) \geq 0, \quad \forall v \in H,$$

which is known as the general mixed variational inequality, see [10].

If K is closed convex set in H and $\varphi(u, v) = I_K(v)$ for all $v \in H$, where I_K is the indicator function of K defined by

$$I_K(v) = \begin{cases} 0, & \text{if } \ v \in K; \\ +\infty, & \text{otherwise.} \end{cases}$$

then the problem (2.1) is equivalent to finding $u^* \in H$ such that $g(u^*) \in K$ and

$$\langle T(u^*), g(v) - g(u^*) \rangle \geq 0, \quad \forall g(v) \in K. \tag{2.2}$$

Problem (2.2) is called the general variational inequality, which was first introduced and studied by Noor [7] in 1988. For the applications, formulation and numerical methods of general variational inequalities (2.2), we refer the reader to the survey [11].

If $g = I$, then the problem (2.2) is equivalent to finding $u^* \in K$ such that

$$\langle T(u^*), v - u^* \rangle \geq 0, \quad \forall v \in K, \tag{2.3}$$

which is known as the classical variational inequality introduced and studied by Stampacchia [16] in 1964. For the applications, formulation and numerical methods, see [1–18].

First, we collect some basic results, which are needed for the rest of the paper.

Definition 2.1 Let $T, g : R^n \rightarrow R^n$ be two mappings. T is said to be g-monotone, if

$$\langle T(v) - T(u), g(v) - g(u) \rangle \geq 0, \quad \forall u, v \in H.$$

Definition 2.2 The bifunction $\varphi(.,.)$ is said to be skew-symmetric, if

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) + \varphi(v, v) \geq 0, \quad \forall u, v \in H. \tag{2.4}$$

Clearly, if the bifunction $\varphi(.,.)$ is linear in both arguments, then,

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) + \varphi(v, v) = \varphi(u - v, u - v) \geq 0, \quad \forall u, v \in H,$$

which shows that the bifunction $\varphi(.,.)$ is nonnegative.

Definition 2.3 ([2]) Let A be a maximal monotone operator, then the resolvent operator associated with A is defined as

$$J_A(u) = (I + \rho A)^{-1}(u), \quad \forall u \in H,$$

where $\rho > 0$ is a constant and I is the identity operator. It is well known that the resolvent operator J_A is single-valued and nonexpansive.

Remark 2.1 It is well known that the subdifferential $\partial\varphi(.,.)$ of a convex , proper and lower-semicontinuous bifunction $\varphi(.,.) : H \times H \longrightarrow R \cup \{+\infty\}$ is a maximal monotone with respect to the first argument, we can define its resolvent by

$$J_{\varphi(u)} = (I + \rho\partial\varphi(., u))^{-1}) \equiv (I + \rho\partial\varphi(u))^{-1}, \tag{2.5}$$

where $\partial\varphi(u) \equiv \partial\varphi(., u)$.

The resolvent operator $J_{\varphi(u)}$defined by (2.5) has the following characterization,

Lemma 2.1 ([10]) *For a given $z \in H$, $u \in H$ satisfies the inequality*

$$\langle u - z, v - u \rangle + \rho\varphi(v, u) - \rho\varphi(u, u) \geq 0, \quad \forall v \in H, \tag{2.6}$$

if and only if

$$u = J_{\varphi(u)}[z],$$

where $J_{\varphi(u)}$ is resolvent operator defined by (2.5).

It follows from Lemma 2.1 that

$$\langle J_{\varphi(u)}[z] - z, v - J_{\varphi(u)}[z]\rangle + \rho\varphi(v, J_{\varphi(u)}[z]) - \rho\varphi(J_{\varphi(u)}[z], J_{\varphi(u)}[z]) \geq 0,$$
$$\forall u, v, z \in H \tag{2.7}$$

The following result can be proved by using Lemma 2.1.

Lemma 2.2 *u^* is solution of problem (2.1) if and only if $u^* \in H$ satisfies the relation:*

$$g(u^*) = J_{\varphi(u^*)}[g(u^*) - \rho T(u^*)], \tag{2.8}$$

where $\rho > 0$ and $J_{\varphi(u)}$ is resolvent operator associated with the bifunction $\varphi(.,.)$.

From Lemma 2.2, it is clear that u is solution of (2.1) if and only if u is a zero point of the residue equation

$$r(u, \rho) := g(u) - J_{\varphi(u)}[g(u) - \rho T(u)] = 0.$$

Throughout this paper, we make following assumptions.

Assumptions:

- H is a finite dimension space.
- g is homeomorphism on H i.e., g is bijective, continuous and g^{-1} is continuous.
- T is continuous and g-monotone operator on H.
- The bifunction $\varphi(.,.)$ is skew-symmetric.
- The solution set of problem (2.1) denoted by S^* is nonempty.

3 Basic results

In this section, we prove some important results, which will be required in our following analysis. The following lemma shows that $\|r(u, \rho)\|$ is a non-decreasing function for $\rho > 0$, to prove this lemma we are using the same techniques as in [1,5].

Lemma 3.1 *For all $u \in H$ and $\rho' > \rho > 0$, it holds that*

$$\|r(u, \rho')\| \geq \|r(u, \rho)\|. \tag{3.1}$$

Proof Let $t := \dfrac{\|r(x, \rho')\|}{\|r(x, \rho)\|}$, using inequality (2.7) we have

$$\begin{aligned}&\langle g(u) - \rho T(u) - J_{\varphi(u)}[g(u) - \rho T(u)], J_{\varphi(u)}[g(u) - \rho T(u)] - J_{\varphi(u)'}[g(u) - \rho' T(u)]\rangle\\ &\quad + \rho\varphi(J_{\varphi(u)'}[g(u) - \rho' T(u)], J_{\varphi(u)}[g(u) - \rho T(u)])\\ &\quad - \rho\varphi(J_{\varphi(u)}[g(u) - \rho T(u)], J_{\varphi(u)}[g(u) - \rho T(u)]) \geq 0,\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}&\langle g(u) - \rho' T(u) - J_{\varphi(u)'}[g(u) - \rho' T(u)], J_{\varphi(u)'}[g(u) - \rho' T(u)] - J_{\varphi(u)}[g(u) - \rho T(u)]\rangle\\ &\quad + \rho'\varphi(J_{\varphi(u)}[g(u) - \rho T(u)], J_{\varphi(u)'}[g(u) - \rho' T(u)])\\ &\quad - \rho'\varphi(J_{\varphi(u)'}[g(u) - \rho' T(u)], J_{\varphi(u)'}[g(u) - \rho' T(u)]) \geq 0,\end{aligned} \tag{3.3}$$

where $J_{\varphi(u)'} = (I + \rho'\partial\varphi(u))^{-1}$, from (3.2) and using

$$J_{\varphi(u)}[g(u) - \rho T(u)] - J_{\varphi(u)'}[g(u) - \rho' T(u)] = r(u, \rho') - r(u, \rho),$$

we obtain

$$\begin{aligned}\langle r(u,\rho), r(u,\rho') - r(u,\rho)\rangle &\geq \rho\langle T(u), r(u,\rho') - r(u,\rho)\rangle \\ &- \rho\varphi(J_{\varphi(u)'}[g(u) - \rho' T(u)], J_{\varphi(u)}[g(u) - \rho T(u)]) \\ &+ \rho\varphi(J_{\varphi(u)}[g(u) - \rho T(u)], J_{\varphi(u)}[g(u) - \rho T(u)]). \end{aligned} \tag{3.4}$$

Similarly, we have

$$\begin{aligned}\langle r(u,\rho'), r(u,\rho) - r(u,\rho')\rangle &\geq \rho'\langle T(u), r(u,\rho) - r(u,\rho')\rangle \\ &- \rho'\varphi(J_{\varphi(u)}[g(u) - \rho T(u)], J_{\varphi(u)'}[g(u) - \rho' T(u)]) \\ &+ \rho'\varphi(J_{\varphi(u)'}[g(u) - \rho' T(u)], J_{\varphi(u)'}[g(u) - \rho' T(u)]). \end{aligned} \tag{3.5}$$

Multiplying (3.4) and (3.5) by ρ' and ρ respectively, and then adding them, using (2.4) we get

$$\langle \rho' r(u,\rho) - \rho r(u,\rho'), r(u,\rho') - r(u,\rho)\rangle \geq 0 \tag{3.6}$$

and consequently

$$\rho'\|r(u,\rho)\|^2 + \rho\|r(u,\rho')\|^2 \leq (\rho+\rho')\langle r(u,\rho), r(u,\rho')\rangle. \tag{3.7}$$

From Cauchy-Schwarz inequality, we have

$$\langle r(u,\rho), r(u,\rho')\rangle \leq \|r(u,\rho)\| \cdot \|r(u,\rho')\|.$$

Then

$$\rho'\|r(u,\rho)\|^2 + \rho\|r(u,\rho')\|^2 \leq (\rho+\rho')\|r(u,\rho)\| \cdot \|r(u,\rho')\|. \tag{3.8}$$

Dividing (3.8) by $\|r(u,\rho)\|^2$ we obtain

$$\rho' + \rho t^2 \leq (\rho + \rho')t$$

which is equivalent to

$$(t-1)\left(t - \frac{\rho'}{\rho}\right) \leq 0, \tag{3.9}$$

this implies $1 \leq t \leq \dfrac{\rho'}{\rho}$, from which the required result follows. □

We end this section by the following lemma which can be regarded as an extension of result of Noor [8]. For the sake of completeness and to convey an idea of the technique involved, we include its proof.

Lemma 3.2 $\forall u \in H$, $u^* \in S^*$ *and* $\rho > 0$, *we have*

$$\langle g(u) - g(u^*) + \rho[T(u) - T(u^*)], r(u,\rho)\rangle \geq \|r(u,\rho)\|^2. \tag{3.10}$$

Proof For any $u^* \in S^*$ solution of problem (2.1), we have

$$\begin{aligned}&\langle \rho T(u^*), g(v) - g(u^*)\rangle + \rho\varphi(g(v), g(u^*)) - \rho\varphi(g(u^*), g(u^*)) \geq 0, \\ &\forall v \in H, \rho > 0. \end{aligned} \tag{3.11}$$

Taking $g(v) = J_{\varphi(u)}[g(u) - \rho T(u)]$ in (3.11), we obtain

$$\begin{aligned}&\langle \rho T(u^*), J_{\varphi(u)}[g(u) - \rho T(u)] - g(u^*)\rangle \\ &\quad + \rho\varphi(J_{\varphi(u)}[g(u) - \rho T(u)], g(u^*)) - \rho\varphi(g(u^*), g(u^*)) \geq 0. \end{aligned} \tag{3.12}$$

Substituting $z = g(u) - \rho T(u)$ and $v = g(u^*)$ into (2.7), and using the definition of $r(u, \rho)$, we get

$$\begin{aligned}\langle r(u,\rho) - \rho T(u), J_{\varphi(u)}[g(u) - \rho T(u)] - g(u^*)\rangle + \rho\varphi(g(u^*), J_{\varphi(u)}[g(u) - \rho T(u)])\\ - \rho\varphi(J_{\varphi(u)}[g(u) - \rho T(u)], J_{\varphi(u)}[g(u) - \rho T(u)]) \geq 0.\end{aligned} \tag{3.13}$$

Adding (3.12) and (3.13), and using the skew-symmetry of the bifunction $\varphi(.,.)$, we have

$$\langle r(u,\rho) - \rho[T(u) - T(u^*], J_{\varphi(u)}[g(u) - \rho T(u)] - g(u^*)\rangle \geq 0,$$

which can be rewritten as

$$\langle r(u,\rho) - \rho[T(u) - T(u^*)], g(u) - g(u^*) - r(u,\rho)\rangle \geq 0,$$

then

$$\begin{aligned}\langle g(u) - g(u^*) + \rho[T(u) - T(u^*)], r(u,\rho)\rangle \geq \|r(u,\rho)\|^2\\ + \rho\langle g(u) - g(u^*), T(u) - T(u^*)\rangle.\end{aligned} \tag{3.14}$$

By using the g-monotonicity of T and (3.14), the conclusion of Lemma 3.2 is proved. □

4 Inexact implicit method and convergence analysis

In this section, we describe the inexact implicit method. First, we need two non-negative sequences $\{\tau_k\}$ and η_k satisfying to the following conditions

$$\sum_{k=0}^{+\infty} \tau_k < +\infty \quad \text{and} \quad \sum_{k=0}^{\infty} \eta_k^2 < +\infty. \tag{4.1}$$

Now, we introduce the inexact implicit method with variable parameter as follows:

Step 0. Given $\epsilon > 0$, $\gamma \in [1, 2)$, $\rho > 0$ and $u^0 \in H$, set $k = 0$.
Step 1. Set $\rho_k = \rho$. If $\|r(u^k, \rho)\| < \epsilon$, then stop, otherwise
Step 2. Find $u^{k+1} \in H$ such that

$$\|\Theta_k(u^{k+1})\| \leq \eta_k \|r(u^k, \rho_k)\|, \tag{4.2}$$

where

$$\Theta_k(u) = g(u) + \rho_k T(u) - \{[g(u^k) + \rho_k T(u^k)] - \gamma r(u^k, \rho_k)\}. \tag{4.3}$$

Step 3. For a given constant $\alpha > 0$, let

$$\omega_k = \frac{\|\rho_k[T(u^{k+1}) - T(u^k)]\|}{\|g(u^{k+1}) - g(u^k)\|},$$

Choose ρ_{k+1} according to the following self-adaptive rule

$$\rho_{k+1} = \begin{cases} (1+\tau_k)\rho_k, & \text{if } \omega_k < \frac{1}{1+\alpha}, \\ \frac{1}{(1+\tau_k)}\rho_k, & \text{if } \omega_k > 1+\alpha, \\ \rho_k, & \text{otherwise.}\end{cases} \tag{4.4}$$

Set $k := k + 1$, and go to step 1.

Remark 4.1 It follows from $\tau_k > 0$ and (4.1) that $\prod_{k=0}^{+\infty}(1+\tau_k) < +\infty$. Denote

$$C_\tau := \prod_{k=0}^{+\infty}(1+\tau_k).$$

Then, $\rho_k \subset [\frac{1}{C_\tau}\rho_0, C_\tau \rho_0]$ is bounded. We let

$$\rho_l := \inf_k\{\rho_k\} > 0 \qquad \text{and} \qquad \rho_u := \sup_k\{\rho_k\} < +\infty.$$

Note that if $\eta_k = 0$ we get the exact solution of $\Theta_k(u) = 0$.

Lemma 4.1 *Let $\{u^k\}$ be the sequence generated by the proposed method. Then for any $u^* \in S^*$, we have*

$$\begin{aligned}&\|g(u^{k+1}) - g(u^*) + \rho_{k+1}[T(u^{k+1}) - T(u^*)]\|^2\\ &\quad \le (1+\tau_k)^2\|g(u^{k+1}) - g(u^*) + \rho_k[T(u^{k+1}) - T(u^*)]\|^2.\end{aligned} \tag{4.5}$$

Proof Since $0 < \rho_{k+1} \le (1+\tau_k)\rho_k$, using the g-monotonicity of T, it follows that

$$\begin{aligned}&\|g(u^{k+1}) - g(u^*) + \rho_{k+1}[T(u^{k+1}) - T(u^*)]\|^2\\ &\quad = \|g(u^{k+1}) - g(u^*)\|^2 + 2\rho_{k+1}\langle g(u^{k+1}) - g(u^*), T(u^{k+1}) - T(u^*)\rangle\\ &\qquad + \rho_{k+1}^2\|T(u^{k+1}) - T(u^*)\|^2\\ &\quad \le \|g(u^{k+1}) - g(u^*)\|^2 + 2(1+\tau_k)\rho_k\langle g(u^{k+1}) - g(u^*), T(u^{k+1}) - T(u^*)\rangle\\ &\qquad + (1+\tau_k)^2\rho_k^2\|T(u^{k+1}) - T(u^*)\|^2\\ &\quad \le (1+\tau_k)^2\{\|g(u^{k+1}) - g(u^*)\|^2 + 2\rho_k\langle g(u^{k+1}) - g(u^*), T(u^{k+1}) - T(u^*)\rangle\\ &\qquad + \rho_k^2\|T(u^{k+1}) - T(u^*)\|^2\}\\ &\quad = (1+\tau_k)^2\|g(u^{k+1}) - g(u^*) + \rho_k[T(u^{k+1}) - T(u^*)]\|^2.\end{aligned} \tag{4.6}$$

We can get the assertion of this lemma. □

The following result plays an important role in the convergence analysis of the proposed method.

Theorem 4.1 *Let $\{u^k\}$ be the sequence generated by the inexact implicit method. Then there exists a $k_0 \ge 0$, such that for all $k \ge k_0$*

$$\begin{aligned}&\|g(u^{k+1}) - g(u^*) + \rho_{k+1}[T(u^{k+1}) - T(u^*)]\|^2\\ &\quad \le (1+\xi_k)\|g(u^k) - g(u^*) + \rho_k[T(u^k) - T(u^*)]\|^2 - c\|r(u^k, \rho_k)\|^2,\end{aligned} \tag{4.7}$$

where

$$\xi_k := (1+\tau_k)^2\left(1 + \frac{4\eta_k^2}{\gamma(2-\gamma)}\right) - 1 \qquad \text{and} \qquad c := \frac{\gamma(2-\gamma)}{2}.$$

Proof It follows from (4.3) that

$$\begin{aligned}
&\|g(u^{k+1}) - g(u^*) + \rho_k[T(u^{k+1}) - T(u^*)]\|^2 \\
&= \|g(u^k) - g(u^*) + \rho_k[T(u^k) - T(u^*)] - [\gamma r(u^k, \rho_k) - \Theta_k(u^{k+1})]\|^2 \\
&= \|g(u^k) - g(u^*) + \rho_k[T(u^k) - T(u^*)]\|^2 + \|\gamma r(u^k, \rho_k) - \Theta_k(u^{k+1})\|^2 \\
&\quad - 2\gamma\langle g(u^k) - g(u^*) + \rho_k[T(u^k) - T(u^*)], r(u^k, \rho_k)\rangle \\
&\quad + 2\langle g(u^k) - g(u^*) + \rho_k[T(u^k) - T(u^*)], \Theta_k(u^{k+1})\rangle \\
&\le \|g(u^k) - g(u^*) + \rho_k[T(u^k) - T(u^*)]\|^2 - 2\gamma\|r(u^k, \rho_k)\|^2 \\
&\quad + 2\langle g(u^k) - g(u^*) + \rho_k[T(u^k) - T(u^*)], \Theta_k(u^{k+1})\rangle \\
&\quad + \|\gamma r(u^k, \rho_k) - \Theta_k(u^{k+1})\|^2, \qquad (4.8)
\end{aligned}$$

where the inequality follows from Lemma 3.2.

Since $\|\Theta_k(u^{k+1})\| \le \eta_k\|r(u^k, \rho_k)\|$ and $\lim_{k\to\infty}\eta_k = 0$, it is easy to show that there is a $k_0 > 0$, such that for all $k \ge k_0$

$$\|\gamma r(u^k, \rho_k) - \Theta_k(u^{k+1})\|^2 \le \gamma^2\|r(u^k, \rho_k)\|^2 + \frac{1}{4}\gamma(2-\gamma)\|r(u^k, \rho_k)\|^2. \qquad (4.9)$$

Using the Cauchy-Schwarz inequality and (4.2), we get

$$\begin{aligned}
&2\langle g(u^k) - g(u^*) + \rho_k[T(u^k) - T(u^*)], \Theta_k(u^{k+1})\rangle \\
&\le \frac{4\eta_k^2}{\gamma(2-\gamma)}\|g(u^k) - g(u^*) + \rho_k[T(u^k) - T(u^*)]\|^2 + \frac{\gamma(2-\gamma)}{4\eta_k^2}\|\Theta_k(u^{k+1})\|^2 \\
&\le \frac{4\eta_k^2}{\gamma(2-\gamma)}\|g(u^k) - g(u^*) + \rho_k[T(u^k) - T(u^*)]\|^2 + \frac{1}{4}\gamma(2-\gamma)\|r(u^k, \rho_k)\|^2. \qquad (4.10)
\end{aligned}$$

Substituting (4.9) and (4.10) into inequality (4.8), we obtain

$$\begin{aligned}
&\|g(u^{k+1}) - g(u^*) + \rho_k[T(u^{k+1}) - T(u^*)]\|^2 \\
&\le \left(1 + \frac{4\eta_k^2}{\gamma(2-\gamma)}\right)\|g(u^k) - g(u^*) + \rho_k[T(u^k) - T(u^*)]\|^2 \\
&\quad - \frac{1}{2}\gamma(2-\gamma)\|r(u^k, \rho_k)\|^2, \qquad (4.11)
\end{aligned}$$

which combined with (4.6) gives the desired result. □

From the above theorem, we get the convergence of the proposed method as follows.

Theorem 4.2 *The sequence* $\{u^k\}$ *generated by the proposed method converges to a solution point of problem* (2.1).

Proof From $\sum_{k=0}^{+\infty}\tau_k < +\infty$ and $\sum_{k=0}^{+\infty}\eta_k^2 < +\infty$ it follows that $\sum_{k=0}^{+\infty}\xi_k < +\infty$ and $\prod_{k=0}^{+\infty}(1+\xi_k) < +\infty$. Denote

$$C_s := \sum_{k=0}^{+\infty}\xi_k < +\infty \quad \text{and} \quad C_p := \prod_{k=0}^{+\infty}(1+\xi_k).$$

Let $\tilde{u} \in S^*$, From (4.7) we get

$$\begin{aligned}
&\|g(u^{k+1}) - g(\tilde{u}) + \rho_{k+1}[T(u^{k+1}) - T(\tilde{u})]\|^2 \\
&\leq \prod_{i=0}^{k}(1+\xi_i)\|g(u^0) - g(\tilde{u}) + \rho_0[T(u^0) - T(\tilde{u})]\|^2 \\
&\leq C_p\|g(u^0) - g(\tilde{u}) + \rho_0[T(u^0) - T(\tilde{u})]\|^2.
\end{aligned}$$

Therefore, there exists a constant $C > 0$ such that

$$\|g(u^k) - g(\tilde{u}) + \rho_k[T(u^k) - T(\tilde{u})]\|^2 \leq C, \qquad \forall k \geq 0. \tag{4.12}$$

From (4.12), g is homeomorphism and the g-monotonicity of T, it is easy to verify that the $\{u^k\}$ is bounded. Combining (4.7) and (4.12), we have

$$\begin{aligned}
c\sum_{k=0}^{+\infty}\|r(u^k, \rho_k)\|^2 &\leq \|g(u^0) - g(\tilde{u}) + \rho_0[T(u^0) - T(\tilde{u})]\|^2 \\
&\quad + \sum_{k=0}^{+\infty}\xi_k\|g(u^k) - g(\tilde{u}) + \rho_k[T(u^k) - T(\tilde{u})]\|^2 \\
&\leq C + C\sum_{k=0}^{+\infty}\xi_k \\
&\leq (1+C_s)C
\end{aligned}$$

and it follows from Lemma 3.1 that

$$\lim_{k\to\infty} r(u^k, \rho_l) = 0.$$

Let u^* be a cluster point of $\{u^k\}$ then there exists a subsequence $\{u_j^k\} \longrightarrow u^*$. Since T and g are continuous then $r(u, \rho_l)$ is continuous, and

$$r(u^*, \rho_l) = \lim_{j\to\infty} r(u^{k_j}, \rho_l) = 0$$

then u^* is a solution point of problem (2.1).

Assume that $\bar{u} \neq u^*$ is another cluster point of $\{u^k\}$. Since u^* is a cluster point of $\{u^k\}$, there exists a $k_0 > 0$ such that

$$\|g(u^{k_0}) - g(u^*) + \rho_{k_0}[T(u^{k_0}) - T(u^*)]\| \leq \frac{1}{2\sqrt{C_p}}\|g(\bar{u}) - g(u^*)\|. \tag{4.13}$$

Using the g- monotonicity of T, for all $k \geq k_0$, we have

$$\begin{aligned}
\|g(u^k) - g(u^*)\| &\leq \|g(u^k) - g(u^*) + \rho_k[T(u^k) - T(u^*)]\| \\
&\leq \left(\prod_{i=k_0}^{k-1}(1+\xi_i)\right)^{\frac{1}{2}} \|g(u^{k_0}) - g(u^*) + \rho_{k_0}[T(u^{k_0}) - T(u^*)]\| \\
&\leq \sqrt{C_p}\|g(u^{k_0}) - g(u^*) + \rho_{k_0}[T(u^{k_0}) - T(u^*)]\| \\
&\leq \frac{1}{2}\|g(\bar{u}) - g(u^*)\|,
\end{aligned}$$

where the second inequality follows from (4.7) and the last inequality follows from (4.13). Then

$$\|g(u^k) - g(\bar{u}) \geq \|g(\bar{u}) - g(u^*)\| - \|g(u^k) - g(u^*)\| \geq \frac{1}{2}\|g(\bar{u}) - g(u^*)\| > 0,$$
$$\forall k \geq k_0.$$

This contradicts with the assumption that $\bar{u}$ is a cluster point of $\{u^k\}$. Thus, it follows immediately that the sequence $\{u^k\}$ has exactly one cluster point, i.e.,

$$\lim_{k\to\infty} u^k = u^*.$$ □

5 Relationship to some existing implicit methods

Some methods can be viewed as special case of our method for example:

- *Method of Noor* [8]. If $g = I, \eta_k = 0$ and $\varphi(v, u^*) = \varphi(v), \forall u^* \in H$, the proposed method reduces to method of Noor [10] with fixed parameter, since the method with self-adaptive parameter is more preferable in practice than the one with fixed parameter. We propose an inexact implicit method with variable parameter.
- *Method of Wang et al.* [17]. If $g = I$ and $\varphi(v, u^*) = \varphi(v), \forall u^* \in H$, the proposed method collapses to method of Wang et al. [17]. The inexact method of Wang et al. proposed to compute η_k explicitly in each iteration which is different to the Step 2 in our method.
- *Method of Zeng and Yao* [18]. If $\varphi(v, u^*) = \varphi(v), \forall u^* \in H$, the proposed method becomes method of Zeng and Yao [18]. We denote by u^{k+1}_{exact} the unique exact solution and u^{k+1} the approximate solution. From 4.2 we have

$$\|g(u^{k+1}) - g(u^{k+1}_{exact})\| \leq \|(g + \rho_k T)(u^{k+1}) - (g + \rho_k T)(u^{k+1}_{exact})\| \leq \eta_k \|r(u^k, \rho_k)\|. \tag{5.14}$$

The inexact version of the method in [18] obtains the approximate solutions u^{k+1} according to the following criteria:

$$\|g(u^{k+1}) - g(u^{k+1}_{exact})\| \leq \|(g + \rho_k T)(u^{k+1}) - (g + \rho_k T)(u^{k+1}_{exact})\| \leq \eta_k, \tag{5.15}$$

where

$$\sum_{k=0}^{\infty} \eta_k < +\infty. \tag{5.16}$$

The main drawback of the inexact criteria (5.15) is that it is difficult to decide wether or not the value η_k is appropriate at a specific iteration. In particular, when the value of η_k is too large, the approximate of solution u^{k+1} may be far away from the exact solution u^{k+1}_{exact}. This fact means that the current iteration make little progress towards the solution. Our new inexact criteria (5.14) make use to control the accuracy of the approximate solution, thus overcome the difficulties caused by (5.15). Furthermore, the criteria (5.16) is relaxed to $\sum_{k=0}^{\infty} \eta_k^2 < +\infty$.

6 Computational results

In this section, we present some numerical results for the proposed inexact implicit method. Our objective is to show that the self-adaptive adjustment rule is necessary in practice.

We consider the following least distance problem:

$$\begin{aligned} \min \quad & \frac{1}{2}\|x-c\|^2 \\ \text{s.t.} \quad & Ax \in \Omega, \end{aligned}$$

where $A \in R^{n\times n}$, $c \in R^n$ and $\Omega \subset R^n$ is a closed convex set. This problem can be written as

$$\begin{aligned} \min \quad & \frac{1}{2}\|x-c\|^2 \\ \text{s.t.} \quad & Ax-\xi=0 \\ & \xi \in \Omega \end{aligned} \tag{6.1}$$

The Lagrangian function of problem (6.1) is

$$L(x,\xi,y)=\frac{1}{2}\langle x,x\rangle-\langle c,x\rangle-\langle y,Ax-\xi\rangle,$$

then

$$L(x^*,\xi^*,y)\le L(x^*,\xi^*,y^*)\le L(x,\xi,y^*),$$

where $(x^*,\xi^*,y^*)\in R^n\times\Omega\times R^n$ is saddle point of the Lagrangian function. From the above inequalities we can obtain $\forall\xi\in\Omega$

$$\begin{cases} x^*=\langle A,y^*\rangle+c; \\ \langle \xi-\xi^*,y^*\rangle\ge 0; \\ Ax^*=\xi^*. \end{cases} \tag{6.2}$$

Substituting the first and the third equation in the second of system (6.2), we get

$$(AA^Ty^*+Ac)\in\Omega, \qquad \langle \xi-(AA^Ty^*+Ac),y^*\rangle \qquad \forall\xi\in\Omega. \tag{6.3}$$

Problem (6.3) is special case of problem (2.1), by taking

$$g(u)=AA^Tu+Ac, \quad T(u)=u \quad \text{and} \quad \varphi(v,u)=\begin{cases} 0, & \text{if } v\in R^n_+; \\ +\infty, & \text{otherwise.} \end{cases}$$

We form the test problem as follows: The matrix A is $n\times n$ matrix whose entries are randomly generalized in the interval $(-5,+5)$, the vector c is generated from a uniform distribution in the interval $(-500, 500)$ and the closed convex set Ω is defined as

$$\Omega := \{z\in R^n \,|\, \|z\|\le a\}.$$

Note that in the case $\|Ac\|>a$, $\|AA^Tu^*+Ac\|=a$ (otherwise $u^*=0$ is the trivial solution). Therefore, we test the problem with $a=\rho\|Ac\|$ and $\rho\in(0,1)$.

We adjust the parameter according the following rule:

$$\rho_{k+1}=\begin{cases} \rho_k*2, & \text{if } \omega_k<\frac{1}{1+\alpha}, \\ \rho_k*0.5, & \text{if } \omega_k>1+\alpha, \\ \rho_k, & \text{otherwise.} \end{cases}$$

We denote (IGMV) and (IGMV$_\rho$) as the proposed inexact implicit methods with fixed and with self-adaptive parameters respectively.

In all tests, we take $\gamma=1.8$, $\alpha=1$, the starting point $u^0=(0,\ldots,0)^T$ and the stopping criterion $\|r(u^k,1)\|_\infty\le 10^{-7}$. All codes were written in Matlab and ran on a P4-2.00GHz notebook computer. The iteration numbers and the computational time of IGMV and IGMV$_\rho$ methods for the problem with different dimensions and initial parameter ρ are given in the following table.

Numerical results for problem (6.3)

ρ	$n = 100$				$n = 300$			
	IGMV method		IGMV$_\rho$ method		IGMV method		IGMV$_\rho$ method	
	No. It.	CPU (s)	No. It.	CPU (s)	No. It.	CPU (s)	No. It.	CPU (s)
10^{-1}	7095	28.87	26	0.65	–	–	32	3.11
10^{-2}	2728	12.17	39	1.22	8112	664.61	48	4.79
10^{-3}	1233	6.86	54	1.54	4419	363.25	61	6.31
10^{-4}	773	6.12	68	1.82	841	74.86	59	6.28
10^{-5}	595	5.22	73	2.30	285	27.70	80	8.35
10^{-6}	408	4.57	64	2.28	211	21.23	94	9.77

– Means that the number of iterations $>$20,000 and CPU time $>$1,000 s

The above table shows that the proposed inexact implicit methods with fixed parameter parameter (IGMV method) depends heavily on the value of ρ. It converges quickly when the parameter ρ is too small. Then IGMV method is problem dependent. Also from this table, we were confirmed that the the proposed inexact implicit methods with self-adaptive parameter (IGMV$_\rho$ method) is very stable and efficient no matter what initial parameter ρ_0 is. In addition, for a set of similar problems, it seems that the iteration numbers are not very sensitive to the problem size.

Acknowledgements Abdellah Bnouhachem was supported partly by NSFC Grants Nos: 70731002 and 70571034, "The Second Term '985' Engineering-Innovation Center for Computational Laboratory of Evolutionary Economic Theory and Its Applications" and "Study on the Evolution of Complex Economic System" at "Innovation Center of Economic Transition and Development of Nanjing University" of Ministry of Education, China.

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